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The Idiot's Guide to the Statistical Theory of Fission Chains

S. Walston

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The Idiot's Guide to the Statistical Theory of Fission Chains

S. Walston trying to understand R. P. Feynman,
F. de Hoffmann, R. Serber, G. I. Bell, W. Hage, D. M. Cifarelli,
N. Snyderman, and M. Prasad

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1 THE BINOMIAL AND POISSON DISTRIBUTIONS[1]

The number of arrangements, or *permutations*, of n objects is $n!$, since the first position can be occupied by any one of the n objects, the second by any of the $(n-1)$ remaining objects, and so on. The number of ordered subsets containing m objects out of n is, by similar reasoning,

$$n(n-1)\cdots(n-m-1) \quad = \quad \frac{n!}{(n-m)!} \tag{1}$$

If we simply ask for the number of subsets containing m objects out of n without regard to the order in which they appear ("number of *combinations* of n objects taken m at a time"), we must divide the above result by $m!$ since each combination may be arranged in $m!$ ways. Thus the number of combinations of n things taken m at a time is the *binomial coefficient*

$$\left(\begin{array}{c} n \\ m \end{array} \right) \quad = \quad \frac{n!}{m!(n-m)!} \tag{2}$$

The result of n successive flips of a fair coin can be represented by a series of letters, each either h or t. For example, $hhth\cdots t$. The probability of any such outcome is $\left(\frac{1}{2}\right)^n$; the number of such arrangements with exactly m heads is again

$$\frac{n!}{m!(n-m)!} \quad = \quad \left(\begin{array}{c} n \\ m \end{array} \right) \tag{3}$$

so that the probability of getting exactly m heads is

$$P_n(m) \quad = \quad \left(\begin{array}{c} n \\ m \end{array} \right) \left(\frac{1}{2}\right)^n \tag{4}$$

Generalizing slightly, we may ask for the probability of exactly m "successes" and $n-m$ "failures" in n repetitions of an experiment, if the probability of a success is p and the probability of a failure is $(1-p)$. The answer is easily found by similar reasoning to be

$$P_n(m) = \binom{n}{m} p^m (1-p)^{n-m} \tag{5}$$

which is the *binomial distribution.* Generalizing further, we can inquire about an experiment with three possible outcomes rather than simply "success" and "failure". Let p_1 be the probability for the first outcome, p_2 the probability for the second, and $1-p_1-p_2$ the probability for the third; let m_1 be the number of instances of the first outcome, m_2 the number of instances of the second, and $n-m_1-m_2$ the number of instances of the third. The probability would then be

$$P_n(m_1, m_2) = \frac{n!}{m_1! m_2! (n-m_1-m_2)!} p_1^{m_1} p_2^{m_2} (1-p_1-p_2)^{n-m_1-m_2} \tag{6}$$

This is the most elementary example of the *multinomial distribution.*

A limiting case of the binomial distribution which is of interest results when $n \to \infty$ and $p \to 0$ in such a way that the product $np = \Lambda$ remains finite. Under the present conditions, with $m \ll n$

$$\frac{n!}{(n-m)!} \to n^m \tag{7}$$

$$(1-p)^{n-m} \to \left(1 - \frac{\Lambda}{n}\right)^n \to e^{-\Lambda} \tag{8}$$

Therefore,

$$P(m; \Lambda) = \frac{\Lambda^m e^{-\Lambda}}{m!} \tag{9}$$

which is the *Poisson distribution.* This is shown in Fig. 1.

Using similar reasoning, we can examine the limiting case of the multinomial distribution which results when $n \to \infty$ and $p_1, p_2 \to 0$ in such a way that $np_1 = \Lambda_1$ and $np_2 = \Lambda_2$ where both remain finite. Under the present conditions, with $m_1, m_2 \ll n$

$$\frac{n!}{(n-m_1-m_2)!} \to n^{m_1} n^{m_2} \tag{10}$$

$$(1-p_1-p_2)^{n-m_1-m_2} \to \left(1 - \frac{\Lambda_1}{n} - \frac{\Lambda_2}{n}\right)^n \to e^{-(\Lambda_1+\Lambda_2)} \tag{11}$$

Therefore,

$$P(m_1, m_2; \Lambda_1, \Lambda_2) = \frac{\Lambda_1^{m_1} \Lambda_2^{m_2} e^{-(\Lambda_1+\Lambda_2)}}{m_1! m_2!} \tag{12}$$

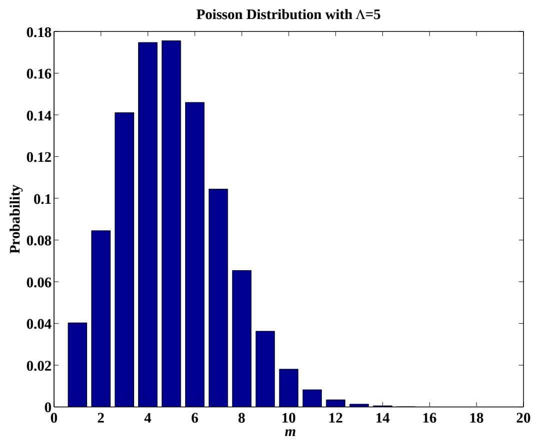

Figure 1: The Poisson distribution gives the probability to count m random events if the expected number is Λ. In this example, $\Lambda = 5$.

2 INTRODUCTION TO TIME DEPENDENCE

The Poisson distribution gives the probability of counting n events over a given amount of time t when the events occur independently of one another (i.e. randomly) at an average rate λ:

$$P_n(t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!} \tag{13}$$

Now consider the probability to count n events over time $t + \Delta t$:

$$P_n(t + \Delta t) = \frac{[\lambda(t + \Delta t)]^n \, e^{-\lambda(t+\Delta t)}}{n!} \tag{14}$$

By using the series expansions for binomials and exponentials, this can be rewritten as

$$\begin{aligned} P_n(t + \Delta t) &= \frac{e^{-\lambda t}}{n!} \\ &\times \left[(\lambda t)^n + n(\lambda t)^{n-1}\lambda\Delta t + \frac{n(n-1)}{2!}(\lambda t)^{n-2}(\lambda\Delta t)^2 + \cdots\right] \\ &\times \left[1 - \lambda\Delta t + \frac{(\lambda\Delta t)^2}{2!} + \cdots\right] \end{aligned} \tag{15}$$

Rearranging in powers of Δt and writing only terms up to first order gives

$$P_n(t + \Delta t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!} - \frac{(\lambda t)^n e^{-\lambda t}}{n!}\lambda\Delta t + \frac{(\lambda t)^{n-1} e^{-\lambda t}}{(n-1)!}\lambda\Delta t + \cdots \tag{16}$$

$$
\begin{aligned}
&= P_n(t) - P_n(t)\lambda\Delta t + P_{n-1}(t)\lambda\Delta t + \cdots && (17)\\
&= P_n(t)(1-\lambda\Delta t) + P_{n-1}(t)\lambda\Delta t + \cdots && (18)
\end{aligned}
$$

The first term on the right may be interpreted as the probability of counting n events over time t and zero events over time Δt; the second term is the probability of counting $n-1$ events over time t and one event over time Δt, and so on.

By rewriting Eq. 18 in a particular way, one can see the definition of the derivative:

$$
\lim_{\Delta t \to 0} \frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = \frac{\partial}{\partial t} P_n(t) = -\lambda P_n(t) + \lambda P_{n-1}(t) \tag{19}
$$

The probability generating function can be constructed by multiplying $P_n(t)$ by x^n and summing over n:

$$
\pi(t,x) = \sum_{n=0}^{\infty} P_n(t) x^n = \sum_{n=0}^{\infty} \frac{(\lambda t)^n e^{-\lambda t}}{n!} x^n \tag{20}
$$

Multiplying Eq. 19 by x^n and summing gives

$$
\begin{aligned}
\frac{\partial}{\partial t} \sum_{n=0}^{\infty} P_n(t) x^n &= -\lambda \sum_{n=0}^{\infty} P_n(t) x^n + \lambda \sum_{n=1}^{\infty} P_{n-1}(t) x^{n-1} x && (21)\\
\frac{\partial \pi}{\partial t} &= (x-1)\lambda\pi && (22)
\end{aligned}
$$

The initial condition that $\pi(0,x) = 1$ is obvious since at $t = 0$, no events would have been counted. The solution to this differential equation is the familiar

$$
\pi(t,x) = e^{(x-1)\lambda t} \tag{23}
$$

The factorial moments of the count distribution are obtained by differentiating this generating function with respect to x and then setting $x = 1$. The first factorial moment is

$$
\begin{aligned}
\left.\frac{\partial \pi}{\partial x}\right|_{x=1} = \lambda t &= \left[\sum_{n=1}^{\infty} n P_n(t) x^{n-1}\right]_{x=1}\\
&= \sum_{n=1}^{\infty} n P_n(t) && (24)
\end{aligned}
$$

and the second factorial moment is

$$
\begin{aligned}
\left.\frac{\partial^2 \pi}{\partial x^2}\right|_{x=1} = (\lambda t)^2 &= \left[\sum_{n=2}^{\infty} n(n-1) P_n(t) x^{n-2}\right]_{x=1}\\
&= \sum_{n=1}^{\infty} n^2 P_n(t) - \sum_{n=1}^{\infty} n P_n(t)\\
&= \sum_{n=1}^{\infty} n^2 P_n(t) - \lambda t && (25)
\end{aligned}
$$

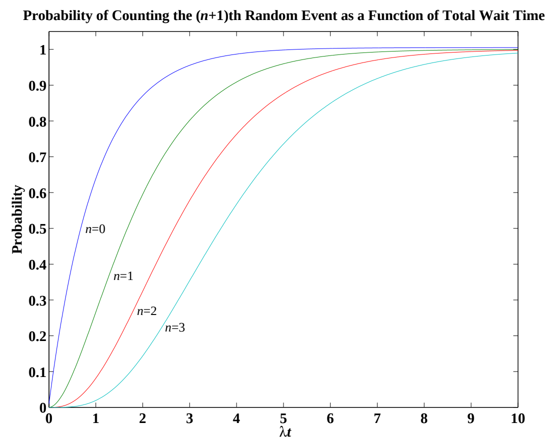

Figure 2: Probability to count the $(n+1)$th (i.e. next) random event as a function of total waiting time t in units of λ^{-1}.

(Note that when $n = 1$, the two terms cancel, so summing from $n = 1$ and summing from $n = 2$ give the same result.)

The variance is also easy to obtain from Eqs. 24 and 25:

$$\sigma^2 = \left[\sum_{n=2}^{\infty} n^2 P_n(t)\right] - \left[\sum_{n=1}^{\infty} n P_n(t)\right]^2 = \left[(\lambda t)^2 + \lambda t\right] - [\lambda t]^2 = \lambda t \tag{26}$$

as expected.

The Poisson distribution, Eq. 13, gives the probability of counting n random events occurring at a rate of λ per unit time during a measurement interval t. The probability of counting the $(n+1)$th random event after waiting an amount of time t is just the cumulative distribution function (CDF) of Eq. 13,

$$\begin{aligned} \mathrm{CDF}(t) &= \lambda \int_0^t \frac{(\lambda x)^n e^{-\lambda x}}{n!} dx \\ &= 1 - \lambda \int_t^{\infty} \frac{(\lambda x)^n e^{-\lambda x}}{n!} dx \\ &= 1 - \frac{\Gamma(n+1, \lambda t)}{n!} \end{aligned} \tag{27}$$

This is shown in Fig. 2. For a process that generates events as a function of time according to a Poisson distribution, the probability distribution of waiting times from an arbitrary starting point (which may be some particular event) to the

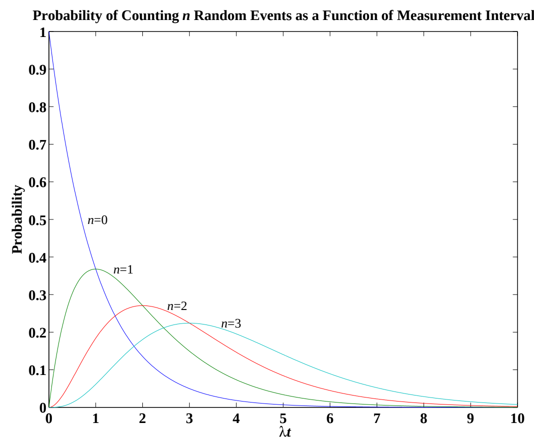

Figure 3: Probability to count $n = k - 1$ events as a function of measurement interval t in units of λ^{-1}.

kth event, where we have defined $k = n + 1$, is then obtained by differentiating the CDF (Eq. 27) with respect to t,

$$\begin{aligned} \frac{d}{dt}\left(1 - \frac{\Gamma\left(k, \lambda t\right)}{(k-1)!}\right) &= \frac{\lambda^k t^{k-1} e^{-\lambda t}}{(k-1)!} \\ &= \frac{\lambda^k t^{k-1} e^{-\lambda t}}{\Gamma(k)} \end{aligned} \quad (28)$$

This is the gamma distribution and is shown in Fig. 3.

3 COUNTING DISTRIBUTIONS FROM FISSION CHAINS

Suppose Λ_1 is the probability to count one neutron from a fission chain, and Λ_2 is the probability to count two neutrons coming from the *same* fission chain. We could ask for the probability of detecting three neutrons each coming from three different fission chains and a single instance of two neutrons coming from the *same* fission chain, for a total of five neutrons. By Eq. 12, the answer would be

$$P(3,1) = \frac{\Lambda_1^3 \Lambda_2 e^{-(\Lambda_1+\Lambda_2)}}{3!} \quad (29)$$

To get the total probability of detecting five neutrons, there are several possibilities which must each be considered: The five neutrons could come from five different fission chains with no cases of two neutrons coming from the same chain; three neutrons could come from three different chains and two from the same chain; or there could be two cases of two neutrons coming from the same chain (for a total of four neutrons) and a fifth neutron from a third chain. All of these possibilities must then be summed:

$$\begin{aligned} b_5 &= P(5,0) + P(3,1) + P(1,2) \\ &= \frac{\Lambda_1^5 e^{-(\Lambda_1+\Lambda_2)}}{5!} + \frac{\Lambda_1^3 \Lambda_2 e^{-(\Lambda_1+\Lambda_2)}}{3!} + \frac{\Lambda_1 \Lambda_2^2 e^{-(\Lambda_1+\Lambda_2)}}{2!} \end{aligned} \tag{30}$$

The number of neutrons which can come from a single fission chain is in principle quite large such that $\Lambda_k \neq 0$ for all k. This results in a more general form for Eq. 12:

$$P(m_1, \cdots, m_n) = \prod_{k=1}^{n} \left(\frac{\Lambda_k^{m_k}}{m_k!} \right) e^{-\left(\sum_{k=1}^{\infty} \Lambda_k\right)}. \tag{31}$$

Now we can ask what the probability is to detect five neutrons:

$$b_5 = \left(\frac{\Lambda_1^5}{5!} + \frac{\Lambda_1^3 \Lambda_2}{3!} + \frac{\Lambda_1 \Lambda_2^2}{2!} + \frac{\Lambda_1^2 \Lambda_3}{2!} + \Lambda_2 \Lambda_3 + \Lambda_1 \Lambda_4 + \Lambda_5 \right) e^{-\left(\sum_{k=1}^{\infty} \Lambda_k\right)} \tag{32}$$

The first three terms are by now familiar; the last four terms, respectively, take into account the instances of two neutrons from two chains and three neutrons from a third chain, two neutrons from one chain and three from another chain, one neutron from one chain and four from another chain, and finally all five neutrons from a single chain. By similar reasoning, the total probability to detect n neutrons is

$$b_n = \sum_{\sum k m_k = n} \left[\prod_{k=1}^{n} \left(\frac{\Lambda_k^{m_k}}{m_k!} \right) \right] e^{-\left(\sum_{k=1}^{\infty} \Lambda_k\right)}. \tag{33}$$

This is the *fission chain probability distribution.*

A useful exercise is to supposes that fission chains are only capable of producing a single neutron at a time, such that $\Lambda_{k>1} = 0$. The total probability b_n to detect n neutrons would then be just

$$b_n = \frac{\Lambda_1^n e^{-\Lambda_1}}{n!} \tag{34}$$

which is once again the ordinary Poisson distribution as expected.

4 THERMAL NEUTRONS FROM A SINGLE FISSION CHAIN

Detecting some number k of neutrons from a single fission chain within a time interval T is the result of a series of events which each has its own probability.

And because independent probabilities multiply, the probability of detecting k neutrons from a single fission chain during the time interval T is simply the product of these probabilities.

Consider a fission chain which produces n neutrons. Of these n neutrons, suppose a total of m actually get detected. Of the m total detected neutrons, only k are detected during the finite time interval T (i.e. $n \geq m \geq k$).

The fission chain neutron multiplicity $\mathcal{P}_n$ is the probability that a fission chain generates n neutrons. The fission chain is assumed to create the neutrons instantaneously (i.e. the neutrons being detected are not fast neutrons). Because there is no upper bound on the length of the fission chain, we must consider all possibilities by summing over n. The details of this quantity will be dealt with in Section 7.

The probability of detecting exactly m neutrons out of a possible n neutrons, if the probability of detection is ϵ, is just the binomial distribution,

$$\mathsf{P}_n(m) \quad = \quad \left(\begin{array}{c} n \\ m \end{array} \right) \epsilon^m (1-\epsilon)^{n-m} \tag{35}$$

Consider now only the m neutrons which ultimately get detected (neutrons which — one way or another — avoid detection were dealt with in Eq. 35). If at time $t = 0$, one starts with m_0 neutrons in the system, and after some time interval dt, m neutrons remain undetected in the system, then $m_0 - m = -dm$ neutrons were detected during dt. Thus, the probability for a neutron to get detected in a time interval dt is

$$\frac{-dm}{m_0} \quad = \quad \lambda\, dt \tag{36}$$

where λ is the neutron detection rate (or alternately λ^{-1} is the neutron lifetime against detection). This differential equation of course has the familiar solution $m(t) = m_0 e^{-\lambda t}$, according to which a neutron created at time s has a probability $e^{-\lambda(t-s)}$ of surviving undetected until time t. The probability of getting detected during the infinitesimal interval dt on the other hand is the product of the probability that the neutron survives from s until t and the probability that it is detected during the interval dt, thus $e^{-\lambda(t-s)}\lambda\, dt$, remembering that independent probabilities multiply. The probability of detecting k out of the total m detected neutrons within the infinitesimal time interval dt is just a binomial distribution in this probability, $e^{-\lambda(t-s)}\lambda\, dt$ (assuming for the moment that all of the n neutrons were created at the same time s),

$$P_m(k) \quad = \quad \left(\begin{array}{c} m \\ k \end{array} \right) \left[e^{-\lambda(t-s)}\lambda\, dt \right]^k \left[1 - e^{-\lambda(t-s)}\lambda\, dt \right]^{m-k}$$

If all of the m neutrons were not created at the same time s, but rather at any time before dt, then we must integrate over s to get the correct probability. Let $\mathbb{R}$ be the rate at which fission chains are initiated. There can be contributions from both spontaneous fission as well as from induced fission. I would prefer to avoid the details of $\mathbb{R}$ for now as it ultimately enters into the combinatorial moments of the count distribution in a non-trivial way. The probability of initiating a fission chain within the time ds is then $\mathbb{R}\,ds$. The probability of detecting k out of the total m detected neutrons within the infinitesimal time interval dt is then

$$P_m(k) \;=\; \left(\begin{array}{c} m \\ k \end{array}\right) \int_{-\infty}^{0} \left[e^{-\lambda(t-s)}\lambda\,dt\right]^k \left[1 - e^{-\lambda(t-s)}\lambda\,dt\right]^{m-k} \mathbb{R}\,ds$$

If the time interval for detection is some finite value T rather than infinitesimal, a further modification becomes necessary:

$$P_m(k) \;=\; \left(\begin{array}{c} m \\ k \end{array}\right) \int_{-\infty}^{0} \left[\int_0^T e^{-\lambda(t-s)}\lambda\,dt\right]^k \left[1 - \int_0^T e^{-\lambda(t-s)}\lambda\,dt\right]^{m-k} \mathbb{R}\,ds$$

This, however, only takes account of the neutrons which got created *before* the time interval T. The probability of detection for neutrons which got created *during* the time interval T is, by similar reasoning

$$P_m(k) \;=\; \left(\begin{array}{c} m \\ k \end{array}\right) \int_{0}^{T} \left[\int_s^T e^{-\lambda(t-s)}\lambda\,dt\right]^k \left[1 - \int_s^T e^{-\lambda(t-s)}\lambda\,dt\right]^{m-k} \mathbb{R}\,ds$$

The total probability to detect k neutrons during a finite time interval T from a total of m detected neutrons, allowing the neutrons to be created at any time before the end of the detection interval, is thus

$$\begin{aligned} P_m(k) \;=\; & \left(\begin{array}{c} m \\ k \end{array}\right) \left\{ \int_{-\infty}^{0} \left[\int_0^T e^{-\lambda(t-s)}\lambda\,dt\right]^k \left[1 - \int_0^T e^{-\lambda(t-s)}\lambda\,dt\right]^{m-k} \mathbb{R}\,ds \right. \\ & \left. + \int_{0}^{T} \left[\int_s^T e^{-\lambda(t-s)}\lambda\,dt\right]^k \left[1 - \int_s^T e^{-\lambda(t-s)}\lambda\,dt\right]^{m-k} \mathbb{R}\,ds \right\} \end{aligned} \qquad (37)$$

There is in principle no limit to the number n of neutrons produced by the fission chain, nor indeed the number m which are ultimately detected except that $n \geq m \geq k$. The probability of detecting k neutrons from a single fission chain within the time interval T is thus the product of $\mathcal{P}_n$, $\mathsf{P}_n(m)$, and $P_m(k)$ summed over the possible combinations of n and m:

$$\begin{aligned} \Lambda_k(T) \;=\; & \sum_{n=k}^{\infty} \mathcal{P}_n \sum_{m=k}^{n} \mathsf{P}_n(m) P_m(k) \\ =\; & \sum_{n=k}^{\infty} \mathcal{P}_n \sum_{m=k}^{n} \left(\begin{array}{c} n \\ m \end{array}\right) \epsilon^m (1-\epsilon)^{n-m} \left(\begin{array}{c} m \\ k \end{array}\right) \\ & \times \left\{ \int_{-\infty}^{0} \left[\int_0^T e^{-\lambda(t-s)}\lambda\,dt\right]^k \left[1 - \int_0^T e^{-\lambda(t-s)}\lambda\,dt\right]^{m-k} \mathbb{R}\,ds \right. \end{aligned}$$

$$+\int_0^T \left[\int_s^T e^{-\lambda(t-s)}\lambda\,dt\right]^k \left[1-\int_s^T e^{-\lambda(t-s)}\lambda\,dt\right]^{m-k} \mathbb{R}\,ds\Bigg\} \tag{38}$$

For the sake of brevity, let us define the following:

$$\eta = \int_0^T e^{-\lambda(t-s)}\lambda\,dt = e^{\lambda s}\left(1-e^{-\lambda T}\right) \tag{39}$$

$$\zeta = \int_s^T e^{-\lambda(t-s)}\lambda\,dt = 1-e^{-\lambda(T-s)} \tag{40}$$

$$\begin{aligned}
\Lambda_k(T) &= \mathbb{R}\sum_{n=k}^{\infty}\mathcal{P}_n\sum_{m=k}^{n}\binom{n}{m}\epsilon^m(1-\epsilon)^{n-m}\binom{m}{k} \\
&\quad\times\left[\int_{-\infty}^{0}\eta^k\,(1-\eta)^{m-k}\;ds+\int_0^T\zeta^k\,(1-\zeta)^{m-k}\;ds\right] \\
&= \mathbb{R}\sum_{n=k}^{\infty}\mathcal{P}_n \\
&\quad\times\left[\int_{-\infty}^{0}\sum_{m=k}^{n}\binom{n}{m}\epsilon^m(1-\epsilon)^{n-m}\binom{m}{k}\eta^k\,(1-\eta)^{m-k}\;ds\right. \\
&\quad\left.+\int_0^T\sum_{m=k}^{n}\binom{n}{m}\epsilon^m(1-\epsilon)^{n-m}\binom{m}{k}\zeta^k\,(1-\zeta)^{m-k}\;ds\right]
\end{aligned} \tag{41}$$

The identity for the product of binomial distributions allows the simplification

$$\sum_{m=k}^{n}\binom{n}{m}\epsilon^m(1-\epsilon)^{n-m}\binom{m}{k}\zeta^k\,(1-\zeta)^{m-k} = \binom{n}{k}(\epsilon\zeta)^k(1-\epsilon\zeta)^{n-k} \tag{42}$$

and similarly for η. The equation for $\Lambda_k(T)$ becomes

$$\begin{aligned}
\Lambda_k(T) &= \mathbb{R}\int_{-\infty}^{0}\sum_{n=k}^{\infty}\mathcal{P}_n\binom{n}{k}(\epsilon\eta)^k(1-\epsilon\eta)^{n-k}\,ds \\
&\quad+\mathbb{R}\int_0^T\sum_{n=k}^{\infty}\mathcal{P}_n\binom{n}{k}(\epsilon\zeta)^k(1-\epsilon\zeta)^{n-k}\,ds
\end{aligned} \tag{43}$$

5 COMBINATORIAL MOMENTS OF THE COUNT DISTRIBUTION

The qth combinatorial moment $\mathcal{M}_q$ of the probability distribution $b_n(T)$ is just the qth factorial moment of $b_n(T)$ divided by $q!$, or more simply

$$\mathcal{M}_q = \sum_{n=q}^{\infty} \binom{n}{q} b_n(T) \tag{44}$$

It will be useful to note that $\mathcal{M}_1 = \bar{c}$, where $\bar{c}$ denotes the average number of counts recorded per unit time. The factorial moments of $b_n(T)$ can be computed from the probability generating function, which can be constructed in the usual way by multiplying $b_n(T)$ by z^n and summing over n, as in Eq. 20. The following algebraic steps show how to write the probability generating function in terms of the Λ_j in a very concise form:

$$\begin{aligned}
\pi(z) &= \sum_{n=0}^{\infty} z^n b_n(T) \qquad (45)\\
&= e^{-\left(\sum_{k=1}^{\infty}\Lambda_k\right)}\left[1 + z\Lambda_1 + z^2\left(\frac{\Lambda_1^2}{2!} + \Lambda_2\right) + z^3\left(\frac{\Lambda_1^3}{3!} + \Lambda_1\Lambda_2 + \Lambda_3\right)\right.\\
&\quad \left. + z^4\left(\frac{\Lambda_1^4}{4!} + \frac{\Lambda_1^2\Lambda_2}{2!} + \Lambda_1\Lambda_3 + \frac{\Lambda_2^2}{2!} + \Lambda_4\right) + \cdots\right]\\
&= e^{-\left(\sum_{k=1}^{\infty}\Lambda_k\right)}\left[1 + z\Lambda_1 + z^2\frac{\Lambda_1^2}{2!} + z^2\Lambda_2 + z^3\frac{\Lambda_1^3}{3!} + z^3\Lambda_1\Lambda_2 + z^3\Lambda_3\right.\\
&\quad \left. + z^4\frac{\Lambda_1^4}{4!} + z^4\frac{\Lambda_1^2\Lambda_2}{2!} + z^4\Lambda_1\Lambda_3 + z^4\frac{\Lambda_2^2}{2!} + z^4\Lambda_4 + \cdots\right]\\
&= e^{-\left(\sum_{k=1}^{\infty}\Lambda_k\right)}\left[\, 1 + \left(z\Lambda_1 + z^2\Lambda_2 + z^3\Lambda_3 + z^4\Lambda_4 + \cdots\right)\right.\\
&\quad + \left(z^2\frac{\Lambda_1^2}{2!} + z^3\Lambda_1\Lambda_2 + z^4\Lambda_1\Lambda_3 + z^4\frac{\Lambda_2^2}{2!} + \cdots\right)\\
&\quad \left. + \left(z^3\frac{\Lambda_1^3}{3!} + z^4\frac{\Lambda_1^2\Lambda_2}{2!} + \cdots\right) + \left(z^4\frac{\Lambda_1^4}{4!} + \cdots\right) + \cdots\right]\\
&= e^{-\left(\sum_{k=1}^{\infty}\Lambda_k\right)}\left[\, 1 + \left(z\Lambda_1 + z^2\Lambda_2 + z^3\Lambda_3 + z^4\Lambda_4 + \cdots\right)\right.\\
&\quad \left. + \frac{1}{2!}\left(z\Lambda_1 + z^2\Lambda_2 + z^3\Lambda_3 + z^4\Lambda_4 + \cdots\right)^2 + \cdots\right]\\
&= e^{-\left(\sum_{k=1}^{\infty}\Lambda_k\right)}\left[1 + \left(\sum_{j=1}^{\infty} z^j\Lambda_j\right) + \frac{1}{2!}\left(\sum_{j=1}^{\infty} z^j\Lambda_j\right)^2 + \cdots\right]\\
&= e^{-\left(\sum_{k=1}^{\infty}\Lambda_k\right)} e^{\left(\sum_{j=1}^{\infty} z^j\Lambda_j\right)}\\
&= e^{\left(\sum_{k=1}^{\infty}(z^k-1)\Lambda_k\right)} \qquad (46)
\end{aligned}$$

The factorial moments are then computed by differentiating Eq. 46 with respect to z. The qth combinatorial moment $\mathcal{M}_q$ is thus

$$\mathcal{M}_q \quad = \quad \frac{1}{q!} \left. \frac{d^q \pi}{dz^q} \right|_{z=1} \tag{47}$$

This enables one to express $\mathcal{M}_q$ in terms of the Λ_k, as follows:

$$\begin{aligned}
\mathcal{M}_1 &= \left. \frac{d\pi}{dz} \right|_{z=1} = \left. \left(\sum_{k=1}^{\infty} k \Lambda_k z^{k-1} \right) e^{\left(\sum_{k=1}^{\infty} (z^k - 1) \Lambda_k \right)} \right|_{z=1} \\
&= \sum_{k=1}^{\infty} k \Lambda_k = \sum_{k=1}^{\infty} \binom{k}{1} \Lambda_k
\end{aligned} \tag{48}$$

$$\mathcal{M}_2 = \sum_{k=2}^{\infty} \binom{k}{2} \Lambda_k + \frac{1}{2!} \left[\sum_{k=1}^{\infty} \binom{k}{1} \Lambda_k \right]^2 \tag{49}$$

$$\begin{aligned}
\mathcal{M}_3 &= \sum_{k=3}^{\infty} \binom{k}{3} \Lambda_k + \left[\sum_{k=2}^{\infty} \binom{k}{2} \Lambda_k \right] \left[\sum_{k=1}^{\infty} \binom{k}{1} \Lambda_k \right] \\
&\quad + \frac{1}{3!} \left[\sum_{k=1}^{\infty} \binom{k}{1} \Lambda_k \right]^3
\end{aligned} \tag{50}$$

We can now define the combinatorial moments of the $\Lambda_k(T)$ as

$$Y_q(T) \quad = \quad \sum_{k=q}^{\infty} \binom{k}{q} \Lambda_k(T) \tag{51}$$

and Eqs. 48, 49, and 50 can be written as

$$\mathcal{M}_1 \quad = \quad Y_1 \quad = \quad \overline{c} \tag{52}$$

$$\mathcal{M}_2 \quad = \quad Y_2 + \frac{\overline{c}^2}{2!} \tag{53}$$

$$\mathcal{M}_3 \quad = \quad Y_3 + Y_2 \overline{c} + \frac{\overline{c}^3}{3!} \tag{54}$$

The analysis of Eqs. 24, 25, and 26 can be extended to derive the variance of the fission chain probability distribution b_n. The variance can be written as

$$\sigma_b^2 \quad = \quad \left[\sum_{n=0}^{\infty} n^2 b_n(T) \right] - \left[\sum_{n=0}^{\infty} n b_n(T) \right]^2 \tag{55}$$

and, analogously to Eqs. 24 and 25,

$$\left. \frac{\partial \pi}{\partial z} \right|_{z=1} \quad = \quad \left[\sum_{n=1}^{\infty} n b_n(T) z^{n-1} \right]_{z=1}$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} n b_n(T) \\
\left.\frac{\partial^2 \pi}{\partial z^2}\right|_{z=1} &= \left[\sum_{n=2}^{\infty} n(n-1) b_n(T) z^{n-2}\right]_{z=1} \\
&= \sum_{n=1}^{\infty} n^2 b_n(T) - \sum_{n=1}^{\infty} n b_n(T)
\end{aligned}
$$

The variance may thus be expressed in terms of derivatives of the probability generating function as

$$
\sigma_b^2 = \left.\frac{\partial^2 \pi}{\partial z^2}\right|_{z=1} + \left.\frac{\partial \pi}{\partial z}\right|_{z=1} - \left[\frac{\partial \pi}{\partial z}\right]_{z=1}^2 \tag{56}
$$

Using the definition of the combinatorial moments from Eq. 47,

$$
\left.\frac{d^q \pi}{dz^q}\right|_{z=1} = q! \mathcal{M}_q \tag{57}
$$

the variance may be expressed as

$$
\begin{aligned}
\sigma_b^2 &= 2\mathcal{M}_2 + \mathcal{M}_1 - \mathcal{M}_1^2 && (58) \\
&= 2Y_2 + \overline{c}^2 + \overline{c} - \overline{c}^2 \\
&= 2Y_2 + \overline{c} && (59)
\end{aligned}
$$

Consider again the combinatorial moments of the $\Lambda_k(T)$ as defined in Eq. 51, and recall the expression for $\Lambda_k(T)$ from Eq. 43:

$$
\begin{aligned}
Y_q(T) &= \sum_{k=q}^{\infty} \binom{k}{q} \Bigg\{ \mathbb{R} \int_{-\infty}^{0} \sum_{n=k}^{\infty} \mathcal{P}_n \binom{n}{k} (\epsilon\eta)^k (1-\epsilon\eta)^{n-k} \, ds \\
&\quad + \mathbb{R} \int_0^T \sum_{n=k}^{\infty} \mathcal{P}_n \binom{n}{k} (\epsilon\zeta)^k (1-\epsilon\zeta)^{n-k} \, ds \Bigg\} \\
&= \mathbb{R} \int_{-\infty}^{0} \sum_{n=k}^{\infty} \mathcal{P}_n \sum_{k=q}^{n} \binom{n}{k} \binom{k}{q} (\epsilon\eta)^k (1-\epsilon\eta)^{n-k} \, ds \\
&\quad + \mathbb{R} \int_0^T \sum_{n=k}^{\infty} \mathcal{P}_n \sum_{k=q}^{n} \binom{n}{k} \binom{k}{q} (\epsilon\zeta)^k (1-\epsilon\zeta)^{n-k} \, ds && (60)
\end{aligned}
$$

This can be simplified further with the identity

$$
\sum_{k=q}^{n} \binom{n}{k} \binom{k}{q} (\epsilon\zeta)^k (1-\epsilon\zeta)^{n-k} = \binom{n}{q} (\epsilon\zeta)^q \tag{61}
$$

thus

$$
Y_q(T) = \mathbb{R} \sum_{n=q}^{\infty} \mathcal{P}_n \binom{n}{q} \epsilon^q \left(\int_{-\infty}^{0} \eta^q \, ds + \int_0^T \zeta^q \, ds \right) \tag{62}
$$

The quantities η and ζ are defined in equations 39 and 40 respectively. The integrals above are then

$$\begin{aligned}
\int_{-\infty}^{0} \eta^q \, ds &= \int_{-\infty}^{0} \left[e^{\lambda s}\left(1 - e^{-\lambda T}\right)\right]^q \, ds \\
&= \frac{\left(1 - e^{-\lambda T}\right)^q}{\lambda q} \qquad (63)\\
\int_0^T \zeta^q \, ds &= \int_0^T \left[1 - e^{-\lambda(T-s)}\right]^q \, ds \qquad (64)\\
&= \begin{cases} \dfrac{-1 + e^{-\lambda T}}{\lambda} + T & q = 1 \\[2ex] -\dfrac{-2\lambda T + e^{-2\lambda T} - 4e^{-\lambda T} + 3}{2\lambda} & q = 2 \\[2ex] -\dfrac{-6\lambda T - 2e^{-3\lambda T} + 9e^{-2\lambda T} - 18e^{-\lambda T} + 11}{6\lambda} & q = 3 \end{cases}
\end{aligned}$$

The term in parenthesis in Eq. 62 becomes

$$\int_{-\infty}^{0} \eta^q \, ds + \int_0^T \zeta^q \, ds = \begin{cases} T & q = 1 \\[2ex] T - \dfrac{1 - e^{-\lambda T}}{\lambda} & q = 2 \\[2ex] T - \dfrac{3 - 4e^{-\lambda T} + e^{-2\lambda T}}{2\lambda} & q = 3 \end{cases} \qquad (65)$$

6 RATE EQUATION FOR THE INTERNAL NEUTRON POPULATION

As a fission chain evolves in time, the neutrons produced by it may

1. do nothing,
2. either get absorbed or leak out of the multiplying medium thus becoming available for detection, or
3. go on to induce subsequent fissions thus perpetuating the chain.

The probability $\mathcal{P}_n$ that a fission chain produces n neutrons that are not absorbed in producing subsequent fissions depends on the probability p that a fission neutron induces a subsequent fission and on the probability distribution C_ν for the fission neutron multiplicity.

If at $t = 0$ there is a *single* neutron in a multiplying medium, the probability that there are n neutrons in said medium at time $t + \Delta t$ can be developed by considering all the possible configurations the system can be in at time t. For example, there could be n neutrons at time t, and nothing happens during the time Δt; there could be $n+1$ neutrons at time t and one could — with probability $1-p$ — get absorbed or leak out during Δt; there could be $n+1-\nu$ neutrons at time t and during Δt one neutron could — with probability p — induce a fission with probability C_ν which produces ν neutrons; there could be $n+2$ neutrons at time t and two could get absorbed or leak out during Δt; and so on. The probability that there are n neutrons in a multiplying medium at time $t+\Delta t$ would then just be the sum of each of these probabilities,

$$\begin{aligned} \mathcal{P}_n(t+\Delta t) &= \mathcal{P}_n(t)\left(1-\frac{\Delta t}{\tau}\right)^n \\ &+ (1-p)\mathcal{P}_{n+1}(t)\,(n+1)\,\frac{\Delta t}{\tau} \\ &+ p\sum_\nu \mathcal{P}_{n+1-\nu}(t)C_\nu\,(n+1-\nu)\,\frac{\Delta t}{\tau} \\ &+ (1-p)^2\mathcal{P}_{n+2}(t)\begin{pmatrix} n+2 \\ 2 \end{pmatrix}\left(\frac{\Delta t}{\tau}\right)^2 \\ &+ \cdots \end{aligned} \tag{66}$$

where τ is the mean neutron lifetime against leakage and absorption (either through gamma conversion or fission). Thus, in the first term, each of the n neutrons has a probability of $(1-\Delta t/\tau)$ of not interacting during the time interval Δt. In the second and third terms, one of the neutrons has a probability $\Delta t/\tau$ of interacting during the interval Δt, and so on. By expanding the first term with the binomial expansion and multiplying through by $\tau/\Delta t$, the derivative can be constructed

$$\lim_{\Delta t\to 0}\tau\frac{\mathcal{P}_n(t+\Delta t)-\mathcal{P}_n(t)}{\Delta t} = \tau\frac{\partial}{\partial t}\mathcal{P}_n(t)$$

and the rate equation is found to be

$$\begin{aligned} \tau\frac{\partial}{\partial t}\mathcal{P}_n(t) &= -\mathcal{P}_n(t)n \\ &+ (1-p)\mathcal{P}_{n+1}(t)\,(n+1) \\ &+ p\sum_\nu \mathcal{P}_{n+1-\nu}(t)C_\nu\,(n+1-\nu) \end{aligned} \tag{67}$$

As in Eq. 20, let the probability generating function and its first derivative with respect to x be

$$f(t,x) = \sum_{n=0}^{\infty}\mathcal{P}_n(t)x^n \tag{68}$$

$$\frac{\partial f}{\partial x} = \sum_{n=1}^{\infty}\mathcal{P}_n(t)nx^{n-1} \tag{69}$$

And as was done with Eq. 21, we can multiply Eq. 67 by x^n and sum to give

$$\begin{aligned}
\tau\frac{\partial}{\partial t}\sum_{n=0}^{\infty}\mathcal{P}_n(t)x^n &= -\sum_{n=1}^{\infty}\mathcal{P}_n(t)nx^{n-1}x \\
&+ \sum_{n=0}^{\infty}(1-p)\mathcal{P}_{n+1}(t)\,(n+1)\,x^n \\
&+ p\sum_{\nu}\sum_{n=\nu}^{\infty}\mathcal{P}_{n+1-\nu}(t)C_\nu\,(n+1-\nu)\,x^{n-\nu}x^\nu \qquad (70) \\
\tau\frac{\partial f}{\partial t} &= \left[-x+(1-p)+p\sum_\nu C_\nu x^\nu\right]\frac{\partial f}{\partial x} \qquad (71)
\end{aligned}$$

Let us define the coefficient in brackets as

$$g(x) = -x+(1-p)+p\sum_\nu C_\nu x^\nu \qquad (72)$$

Its first derivative with respect to x is then

$$g'(x) = -1+p\sum_\nu \nu C_\nu x^{\nu-1} \qquad (73)$$

Note that $g(1)=0$ (because C_ν is assumed to be properly normalized), $g'(1) = -1+p\overline{\nu}$ where

$$\begin{aligned}
\overline{\nu} &= \sum_\nu \nu C_\nu \qquad (74) \\
k_{\mathrm{eff}} &= p\overline{\nu} \qquad (75)
\end{aligned}$$

The first moment of Eq. 71 is, by the product rule,

$$\begin{aligned}
\tau\left.\frac{\partial^2 f}{\partial t\partial x}\right|_{x=1} &= g'(x)\left.\frac{\partial f}{\partial x}\right|_{x=1} + g(x)\left.\frac{\partial^2 f}{\partial x^2}\right|_{x=1} \qquad (76) \\
\tau\frac{\partial}{\partial t}\sum_{n=1}^{\infty}\mathcal{P}_n(t)n &= (-1+p\overline{\nu})\sum_{n=1}^{\infty}\mathcal{P}_n(t)n \qquad (77)
\end{aligned}$$

This differential equation has the familiar solution

$$\sum_{n=1}^{\infty}\mathcal{P}_n(t)n = e^{-\frac{t}{\tau}(1-k_{\mathrm{eff}})} \qquad (78)$$

This is the probability that a neutron which was created at time $t=0$ survives until time t. In the absence of fission, it would just be $e^{-t/\tau}$ with the decay probability being determined only by the neutron lifetime. With fission, the neutron population is replenished with the consequence that the survival probability decays more slowly.

The probability that a neutron induces a fission between times t_f and $t_\mathrm{f}+dt$ is dt/τ_f where τ_f is the neutron lifetime against fission and is longer than the total neutron lifetime by a factor of $1/p$, i.e. $p = \tau/\tau_\mathrm{f}$. And because independent probabilities multiply, the probability that a neutron created at time $t = 0$ survives until time t_f, and then induces a fission between times t_f and $t_\mathrm{f} + dt$ is just $e^{-\frac{t}{\tau}(1-k_\mathrm{eff})}\, dt/\tau_\mathrm{f}$. The total number of induced fissions is on average is then

$$\begin{aligned} \int_0^\infty e^{-\frac{t}{\tau}(1-k_\mathrm{eff})}\, dt/\tau_\mathrm{f} &= \frac{\tau}{\tau_\mathrm{f}\,(1-k_\mathrm{eff})} \\ &= \frac{p}{(1-k_\mathrm{eff})} \end{aligned} \tag{79}$$

The system multiplication can be defined as

$$M = \frac{1}{(1-k_\mathrm{eff})} \tag{80}$$

and using the definition of k_eff from Eq. 75

$$\frac{p}{(1-k_\mathrm{eff})} = \frac{M-1}{\overline{\nu}} \tag{81}$$

7 RATE EQUATION FOR THE FISSION CHAIN NEUTRON POPULATION

The analysis of Section 6 can be extended by considering the probability that, at time $t+\Delta t$, there are m neutrons in the multiplying medium <u>and</u> n neutrons which have left the medium (either through non-fission absorption or leakage). Eq. 66 becomes

$$\begin{aligned} \mathcal{P}_{m,n}(t+\Delta t) &= \mathcal{P}_{m,n}(t)\left(1-\frac{\Delta t}{\tau}\right)^m \\ &+ (1-p)\mathcal{P}_{m+1,n-1}(t)\,(m+1)\,\frac{\Delta t}{\tau} \\ &+ p\sum_\nu \mathcal{P}_{m+1-\nu,n}(t)C_\nu\,(m+1-\nu)\,\frac{\Delta t}{\tau} \\ &+ (1-p)^2\mathcal{P}_{m+2,n-2}(t)\begin{pmatrix} m+2 \\ 2 \end{pmatrix}\left(\frac{\Delta t}{\tau}\right)^2 \\ &+ \cdots \end{aligned} \tag{82}$$

The corresponding rate equation is found to be

$$\begin{aligned} \tau\frac{\partial}{\partial t}\mathcal{P}_{m,n}(t) &= -\mathcal{P}_{m,n}(t)m \\ &+ (1-p)\mathcal{P}_{m+1,n-1}(t)\,(m+1) \\ &+ p\sum_\nu \mathcal{P}_{m+1-\nu,n}(t)C_\nu\,(m+1-\nu) \end{aligned} \tag{83}$$

The probability generating function then becomes

$$f(t,x,y) \;=\; \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\mathcal{P}_{m,n}(t)x^m y^n \tag{84}$$

Analogously to Eq. 71, we can multiply Eq. 83 by $x^m y^n$ and sum over m and n,

$$\begin{aligned}
\tau\frac{\partial}{\partial t}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\mathcal{P}_{m,n}(t)x^m y^n \;&=\; -\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\mathcal{P}_{m,n}(t)mx^{m-1}xy^n \\
&+\; \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(1-p)\mathcal{P}_{m+1,n-1}(t)\,(m+1)\,x^m y^{n-1}y \\
&+\; p\sum_{\nu}\sum_{m=\nu}^{\infty}\sum_{n=0}^{\infty}\mathcal{P}_{m+1-\nu,n}(t)C_\nu\,(m+1-\nu)\,x^{m-\nu}x^\nu y^n
\end{aligned} \tag{85}$$

which can be written more simply as

$$\tau\frac{\partial f}{\partial t} \;=\; \left[-x+(1-p)y+p\sum_{\nu}C_\nu x^\nu\right]\frac{\partial f}{\partial x} \tag{86}$$

Define the coefficient in brackets as

$$g(x,y) \;=\; -x+qy+p\,\mathbb{C}(x) \tag{87}$$

where

$$\mathbb{C}(x) \;=\; \sum_{\nu}C_\nu x^\nu \tag{88}$$

$$q \;=\; 1-p \tag{89}$$

Now consider the quantity $G(x,y)$ such that

$$\begin{aligned}
\frac{\partial f}{\partial G} \;&=\; g(x,y)\frac{\partial f}{\partial x} \\
\frac{\partial G}{\partial x}\frac{\partial f}{\partial G} \;&=\; g(x,y)\frac{\partial G}{\partial x}\frac{\partial f}{\partial x} \\
\frac{\partial f}{\partial x} \;&=\; g(x,y)\frac{\partial G}{\partial x}\frac{\partial f}{\partial x} \\
\frac{\partial G}{\partial x} \;&=\; \frac{1}{g(x,y)} \\
G(x,y) \;&=\; \int\frac{dx}{g(x,y)}
\end{aligned} \tag{90}$$

Eq. 86 can now be written as

$$\frac{\partial f}{\partial(t/\tau)} \;=\; \frac{\partial f}{\partial G} \tag{91}$$

For this identity to hold, differentiating f with respect to t/τ and differentiating f with respect to G must have exactly the same behavior. The only way this can attain is if

$$f(t,x,y) = f\left[\frac{t}{\tau} + G(x,y)\right] \tag{92}$$

Making the assumption that at $t = 0$ there is exactly one neutron in the system, and the further assumption that none have yet leaked out, then

$$\mathcal{P}_{1,0}(0) = 1 \tag{93}$$
$$\mathcal{P}_{n\neq 1,m>0}(0) = 0 \tag{94}$$

and by Eq. 84

$$f(0,x,y) = x \tag{95}$$

Because the y-dependence has dropped out, when combined with Eq. 92, we find that

$$x = f\left[G(x)\right] = G^{-1}\left[G(x)\right] \tag{96}$$

and thus

$$f = G^{-1} \tag{97}$$

In the asymptotic limit

$$\lim_{t\to\infty} G\left\{f\left[\frac{t}{\tau} + G(x,y)\right]\right\} \rightarrow \infty \tag{98}$$

For the subcritical systems of interest here, as $t \to \infty$, there are no neutrons left in the system and all of the neutrons have leaked out. As a result, $m \to 0$ and by Eq. 84 the x-dependence drops out, i.e.

$$\lim_{t\to\infty} f(t,x,y) \rightarrow h(y) \tag{99}$$

But in any case

$$G\left[h(y)\right] = \int \frac{dx}{g\left[h(y)\right]} \rightarrow \infty \tag{100}$$

This is achieved by

$$g\left[h(y)\right] = 0 = -h(y) + qy + p\,\mathbb{C}\left[h(y)\right] \tag{101}$$

8 COMBINATORIAL MOMENTS OF THE FISSION CHAIN NEUTRON MULTIPLICITY DISTRIBUTION

The qth combinatorial moment of the fission chain neutron multiplicity distribution $\mathcal{P}_n$ is just the qth factorial moment of $\mathcal{P}_n$ divided by $q!$. The factorial moments are in turn computed by differentiating Eq. 101 with respect to y. In Eq. 93, we assumed there was one neutron in the system at $t = 0$. This implies that, for the probability generating function of Eq. 101, the first fission is an induced fission. For fission chains initiated by induced fission, the first three factorial moments are thus

$$\begin{aligned} h'(y) &= (1-p) + p\,\mathbb{C}'[h(y)]h'(y) \\ &= \frac{1-p}{1-p\,\mathbb{C}'[h(y)]} \end{aligned} \tag{102}$$

$$\begin{aligned} h''(y) &= p\,\mathbb{C}''[h(y)]\,[h'(y)]^2 + p\,\mathbb{C}'[h(y)]h''(y) \\ &= \frac{p\,\mathbb{C}''[h(y)]\,[h'(y)]^2}{1-p\,\mathbb{C}'[h(y)]} \end{aligned} \tag{103}$$

$$\begin{aligned} h'''(y) &= p\,\mathbb{C}'''[h(y)]\,[h'(y)]^3 + 3p\,\mathbb{C}''[h(y)]h'(y)h''(y) + p\,\mathbb{C}'[h(y)]h'''(y) \\ &= \frac{p\,\mathbb{C}'''[h(y)]\,[h'(y)]^3 + 3p\,\mathbb{C}''[h(y)]h'(y)h''(y)}{1-p\,\mathbb{C}'[h(y)]} \end{aligned} \tag{104}$$

From Eqs. 84 and 99, it can be seen that

$$h(y) = \sum_{n=0}^{\infty} \mathcal{P}_n(t) y^n \tag{105}$$

and because $\mathcal{P}_n$ is assumed to be properly normalized, $h(1) = 1$. From Eq. 88,

$$\begin{aligned} \mathbb{C}'(x) &= \sum_\nu \nu C_\nu x^{\nu-1} \\ \mathbb{C}'(1) &= \sum_\nu \nu C_\nu = \nu_{(1)} = \overline{\nu} \end{aligned} \tag{106}$$

$$\begin{aligned} \mathbb{C}''(x) &= \sum_\nu \nu(\nu-1) C_\nu x^{\nu-2} \\ \mathbb{C}''(1) &= \sum_\nu \nu(\nu-1) C_\nu = \nu_{(2)} \end{aligned} \tag{107}$$

$$\begin{aligned} \mathbb{C}'''(x) &= \sum_\nu \nu(\nu-1)(\nu-2) C_\nu x^{\nu-3} \\ \mathbb{C}'''(1) &= \sum_\nu \nu(\nu-1)(\nu-2) C_\nu = \nu_{(3)} \end{aligned} \tag{108}$$

Applying Eq. 106 to Eq. 102, we have

$$h'(1) = \frac{1-p}{1-p\overline{\nu}} = M_\mathrm{e} \tag{109}$$

where M_e is variously known as the escape multiplication or leakage multiplication of the system. It is useful to note that

$$\frac{p}{1-p\overline{\nu}} = \frac{M_\mathrm{e}-1}{\overline{\nu}-1} = \frac{M-1}{\overline{\nu}} \tag{110}$$

Setting $y=1$ in Eq. 103 and applying Eqs. 106, 107, 109, and 110, we have

$$\begin{aligned} h''(1) &= \frac{p\nu_{(2)}M_\mathrm{e}^2}{1-p\overline{\nu}} \\ &= M_\mathrm{e}^2\frac{M_\mathrm{e}-1}{\overline{\nu}-1}\nu_{(2)} \end{aligned} \tag{111}$$

Setting $y=1$ in Eq. 104 and applying Eqs. 106, 107, 108, 109, and 110, we have

$$h'''(1) = \frac{p\nu_{(3)}M_\mathrm{e}^3}{1-p\overline{\nu}} + \frac{3p\nu_{(2)}M_\mathrm{e}}{1-p\overline{\nu}}\frac{p\nu_{(2)}M_\mathrm{e}^2}{1-p\overline{\nu}} \tag{112}$$

$$= M_\mathrm{e}^3\frac{M_\mathrm{e}-1}{\overline{\nu}-1}\nu_{(3)} + 3M_\mathrm{e}^3\left(\frac{M_\mathrm{e}-1}{\overline{\nu}-1}\right)^2\nu_{(2)}^2 \tag{113}$$

The corresponding combinatorial moments of $\mathcal{P}_n$ for chains initiated by induced fission are

$$\sum_{n=1}^{\infty}\mathcal{P}_n\begin{pmatrix} n \\ 1 \end{pmatrix} = \frac{h'(1)}{1!} = M_\mathrm{e} \tag{114}$$

$$\sum_{n=2}^{\infty}\mathcal{P}_n\begin{pmatrix} n \\ 2 \end{pmatrix} = \frac{h''(1)}{2!} = M_\mathrm{e}^2\frac{M_\mathrm{e}-1}{\overline{\nu}-1}\frac{\nu_{(2)}}{2!} = M_\mathrm{e}^2\frac{M_\mathrm{e}-1}{\overline{\nu}-1}\overline{\nu_2} \tag{115}$$

$$\begin{aligned} \sum_{n=3}^{\infty}\mathcal{P}_n\begin{pmatrix} n \\ 3 \end{pmatrix} &= \frac{h'''(1)}{3!} \\ &= M_\mathrm{e}^3\frac{M_\mathrm{e}-1}{\overline{\nu}-1}\frac{\nu_{(3)}}{3!} + M_\mathrm{e}^3\frac{3}{3!}\left(\frac{M_\mathrm{e}-1}{\overline{\nu}-1}\right)^2(2!)^2\left(\frac{\nu_{(2)}}{2!}\right)^2 \\ &= M_\mathrm{e}^3\frac{M_\mathrm{e}-1}{\overline{\nu}-1}\overline{\nu_3} + 2M_\mathrm{e}^3\left(\frac{M_\mathrm{e}-1}{\overline{\nu}-1}\right)^2\overline{\nu_2}^2 \qquad (116) \\ &= M_\mathrm{e}^3\frac{M_\mathrm{e}-1}{\overline{\nu}-1}\left[\overline{\nu_3} + 2\frac{M_\mathrm{e}-1}{\overline{\nu}-1}\overline{\nu_2}^2\right] \qquad (117) \end{aligned}$$

where

$$\overline{\nu_\mu} = \sum_{\nu=\mu}^{\infty}\begin{pmatrix} \nu \\ \mu \end{pmatrix}C_\nu = \frac{\nu_{(\mu)}}{\mu!} \tag{118}$$

are the combinatorial moments of the neutron multiplicity distribution for induced fission. The two terms in Eq. 116 correspond to the three diagrams in

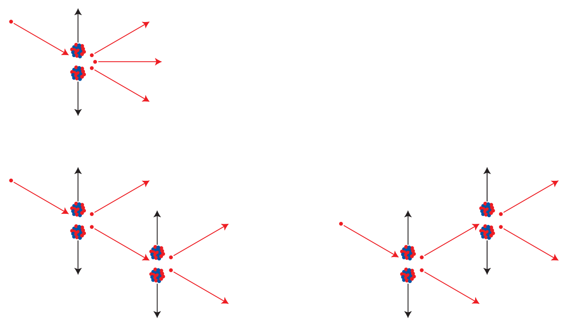

Figure 4: Distinct chains which produce three neutrons starting from an induced fission.

Fig. 4 (the two lower diagrams contain the same factors which accounts for the 2 multiplying the second term).

The diagrams in Fig. 4 suggest something analogous to Feynman diagrams, and they can be drawn more stylistically as in Fig. 7. We define $\rightarrow$ to represent a neutron which does not induce another fission and is thus available for detection and $\bigotimes$ to represent an induced fission. As with Feynman diagrams, there are rules which translate the diagrams into mathematics. By comparing Eq. 116 with Fig. 7, it can be seen that the rules are as follows:

$$\rightarrow \quad = \quad M_{\mathrm{e}} \tag{119}$$

$$\bigotimes \quad = \quad \frac{M_{\mathrm{e}}-1}{\overline{\nu}-1}\overline{\nu_\mu} \tag{120}$$

where μ is the number of lines coming out of the fission. And as with Feynman diagrams, the final result is the sum over all diagrams. Figs. 5, 6 and 7 can be seen to correspond to Eqs. 114, 115, and 116 respectively.

Including chains initiated by a spontaneous fission complicates the situation somewhat, however the rules in Eqs. 119 and 120 still apply. To these must be added one additional rule: Let $\bigodot$ represent a spontaneous fission. Spontaneous fission is the same as induced fission except that there is no factor $\frac{M_{\mathrm{e}}-1}{\overline{\nu}-1}$ since there is no incoming neutron, and the combinatorial moments of the neutron multiplicity distribution for spontaneous fission,

$$\overline{\nu_{\mathrm{S}\mu}} \quad = \quad \sum_{\nu=\mu}^{\infty} \begin{pmatrix} \nu \\ \mu \end{pmatrix} C_{\mathrm{S}\nu} \tag{121}$$

must be used instead of those for induced fission from Eq. 118. Thus,

$$\bigodot \quad = \quad \overline{\nu_{\mathrm{S}\mu}} \tag{122}$$

Applying these rules to the diagrams in Figs. 8, 9, and 10, we can determine the

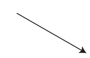

Figure 5: An induced fission which produces one neutron is indistinguishable from a single neutron doing nothing.

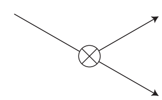

Figure 6: Distinct chain which produces two neutrons starting from an induced fission.

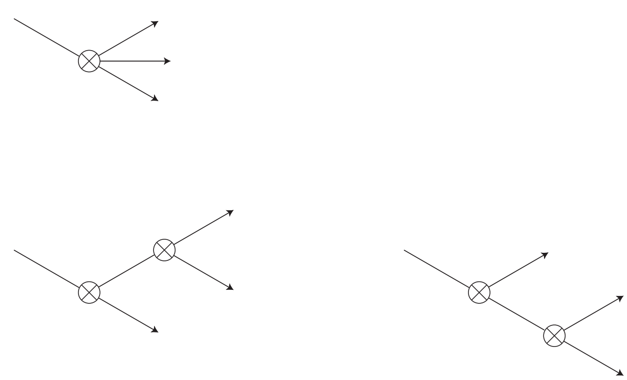

Figure 7: Distinct chains which produce three neutrons starting from an induced fission.

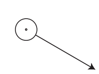

Figure 8: Distinct chain which produces one neutron starting from a spontaneous fission.

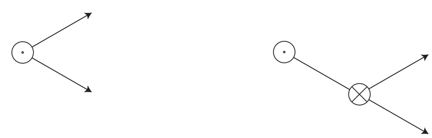

Figure 9: Distinct chains which produce two neutrons starting from a spontaneous fission.

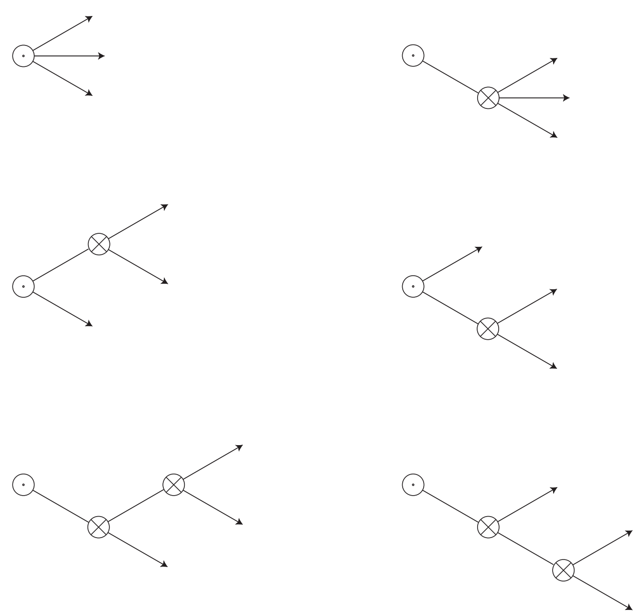

Figure 10: Distinct chains which produce three neutrons starting from a spontaneous fission.

combinatorial moments of the fission chain neutron multiplicity distribution $\mathcal{P}_{\mathrm{S}n}$ for chains initiated by a spontaneous fission. The corresponding combinatorial moments of $\mathcal{P}_{\mathrm{S}n}$ are

$$\sum_{n=1}^{\infty} \mathcal{P}_{\mathrm{S}n} \binom{n}{1} = M_{\mathrm{e}} \overline{\nu_{\mathrm{S}1}} \tag{123}$$

$$\sum_{n=2}^{\infty} \mathcal{P}_{\mathrm{S}n} \binom{n}{2} = M_{\mathrm{e}}^2 \overline{\nu_{\mathrm{S}2}} + M_{\mathrm{e}}^2 \overline{\nu_{\mathrm{S}1}} \frac{M_{\mathrm{e}}-1}{\overline{\nu}-1} \overline{\nu_2} \tag{124}$$

$$= M_{\mathrm{e}}^2 \left[\overline{\nu_{\mathrm{S}2}} + \frac{M_{\mathrm{e}}-1}{\overline{\nu}-1} \overline{\nu_{\mathrm{S}1}}\ \overline{\nu_2} \right] \tag{125}$$

$$\sum_{n=3}^{\infty} \mathcal{P}_{\mathrm{S}n} \binom{n}{3} = M_{\mathrm{e}}^3 \overline{\nu_{\mathrm{S}3}} + M_{\mathrm{e}}^3 \overline{\nu_{\mathrm{S}1}} \frac{M_{\mathrm{e}}-1}{\overline{\nu}-1} \overline{\nu_3} + 2 M_{\mathrm{e}}^3 \overline{\nu_{\mathrm{S}2}} \frac{M_{\mathrm{e}}-1}{\overline{\nu}-1} \overline{\nu_2}$$

$$+ 2 M_{\mathrm{e}}^3 \overline{\nu_{\mathrm{S}1}} \left(\frac{M_{\mathrm{e}}-1}{\overline{\nu}-1} \right)^2 \overline{\nu_2}^2 \tag{126}$$

$$= M_{\mathrm{e}}^3 \left[\overline{\nu_{\mathrm{S}3}} + \frac{M_{\mathrm{e}}-1}{\overline{\nu}-1} \left(\overline{\nu_{\mathrm{S}1}}\ \overline{\nu_3} + 2 \overline{\nu_{\mathrm{S}2}}\ \overline{\nu_2} \right) \right.$$

$$\left. + 2 \left(\frac{M_{\mathrm{e}}-1}{\overline{\nu}-1} \right)^2 \overline{\nu_{\mathrm{S}1}}\ \overline{\nu_2}^2 \right] \tag{127}$$

Let us now reconsider the factor $\mathbb{R}$ from, most notably, Eq. 62 among others. Let

$$\mathbb{R} = \begin{cases} F_{\mathrm{I}} & \text{Induced fission rate} \\ F_{\mathrm{S}} & \text{Spontaneous fission rate} \end{cases} \tag{128}$$

where F_{I} is a function of the neutron flux either from external sources or from within the object, e.g. (α, n). We can define the leading term on the right hand side of Eq. 62 as

$$R_q = F_{\mathrm{I}} \sum_{n=q}^{\infty} \mathcal{P}_n \binom{n}{q} + F_{\mathrm{S}} \sum_{n=q}^{\infty} \mathcal{P}_{\mathrm{S}n} \binom{n}{q} \tag{129}$$

Summing the contributions from chains initiated by both induced and spontaneous fission, we find that

$$R_q = \begin{cases} F_{\mathrm{I}} M_{\mathrm{e}} + F_{\mathrm{S}} M_{\mathrm{e}} \overline{\nu_{\mathrm{S1}}} & q = 1 \\[2ex] F_{\mathrm{I}} M_{\mathrm{e}}^2 \dfrac{M_{\mathrm{e}} - 1}{\overline{\nu} - 1} \overline{\nu_2} & \\ \quad + F_{\mathrm{S}} M_{\mathrm{e}}^2 \left[\overline{\nu_{\mathrm{S2}}} + \dfrac{M_{\mathrm{e}} - 1}{\overline{\nu} - 1} \overline{\nu_{\mathrm{S1}}}\, \overline{\nu_2} \right] & q = 2 \\[2ex] F_{\mathrm{I}} M_{\mathrm{e}}^3 \dfrac{M_{\mathrm{e}} - 1}{\overline{\nu} - 1} \left[\overline{\nu_3} + 2 \dfrac{M_{\mathrm{e}} - 1}{\overline{\nu} - 1} \overline{\nu_2}^2 \right] & \\ \quad + F_{\mathrm{S}} M_{\mathrm{e}}^3 \left[\overline{\nu_{\mathrm{S3}}} + \dfrac{M_{\mathrm{e}} - 1}{\overline{\nu} - 1} \left(\overline{\nu_{\mathrm{S1}}}\, \overline{\nu_3} + 2 \overline{\nu_{\mathrm{S2}}}\, \overline{\nu_2} \right) \right. & \\ \qquad \left. + 2 \left(\dfrac{M_{\mathrm{e}} - 1}{\overline{\nu} - 1} \right)^2 \overline{\nu_{\mathrm{S1}}}\, \overline{\nu_2}^2 \right] & q = 3 \end{cases} \tag{130}$$

9 EXTRACTING USEFUL INFORMATION FROM THE COUNT DISTRIBUTIONS

Let N be the number of time intervals of duration T which are examined, and let $B_n(T)$ be the number of those time intervals in which n neutrons were detected. So in other words, suppose that during the first time interval, six neutrons were counted; B_6 would be incremented by one. During the next time interval, say eight neutrons were counted; B_8 would be incremented by one, and so on for all N time intervals. In this way, the count distribution $B_n(T)$ is built up. The probability distribution $b_n \approx B_n/N$ is just the probability of counting n neutrons during a time interval of duration T. The total number of neutrons counted during all N time intervals is

$$n_{\mathrm{Total}} = \sum_{n=0}^{\infty} n B_n \tag{131}$$

As a practical matter, the combinatorial moments $\mathcal{M}_q$ of the distribution $b_n(T)$ are easy to compute. From Eqs. 52, 53, and 54, the quantities Y_q can be expressed in terms of $\mathcal{M}_q$ as

$$Y_1 = \mathcal{M}_1 = \overline{c} = \frac{n_{\mathrm{Total}}}{N} \tag{132}$$

$$Y_2 = \mathcal{M}_2 - \frac{\overline{c}^2}{2!} \tag{133}$$

$$\begin{aligned} Y_3 &= \mathcal{M}_3 - Y_2 \overline{c} - \frac{\overline{c}^3}{3!} \\ &= \mathcal{M}_3 - \mathcal{M}_2 \overline{c} + \frac{\overline{c}^3}{3} \end{aligned} \tag{134}$$

By applying Eqs. 65 and 129 to Eq. 62 we also know that

$$Y_1 = R_1 \epsilon T \tag{135}$$

$$Y_2 = R_2 \epsilon^2 \left(T - \frac{1 - e^{-\lambda T}}{\lambda} \right) \tag{136}$$

$$Y_3 = R_3 \epsilon^3 \left(T - \frac{3 - 4e^{-\lambda T} + e^{-2\lambda T}}{2\lambda} \right) \tag{137}$$

It is convenient to define the following:

$$Y_{2F} = \frac{Y_2}{Y_1} \tag{138}$$

$$R_{2F} = \frac{R_2}{R_1} \tag{139}$$

$$Y_{3F} = \frac{Y_3}{Y_1} \tag{140}$$

$$R_{3F} = \frac{R_3}{R_1} \tag{141}$$

Applying these definitions to Eqs. 136 and 137, the quantities normally used in the analysis are found to be

$$Y_{2F} = R_{2F} \epsilon \left(1 - \frac{1 - e^{-\lambda T}}{\lambda T} \right) \tag{142}$$

$$Y_{3F} = R_{3F} \epsilon^2 \left(1 - \frac{3 - 4e^{-\lambda T} + e^{-2\lambda T}}{2\lambda T} \right) \tag{143}$$

or, as expressed in terms of the combinatorial moments of the count distributions,

$$Y_{2F} = \frac{\mathcal{M}_2}{\bar{c}} - \frac{\bar{c}}{2!} \tag{144}$$

$$Y_{3F} = \frac{\mathcal{M}_3}{\bar{c}} - \mathcal{M}_2 + \frac{\bar{c}^2}{3} \tag{145}$$

It is worth noting that

$$\lim_{\lambda T \to 0} Y_{2F} = 0 \tag{146}$$

$$\lim_{\lambda T \to 0} Y_{3F} = 0 \tag{147}$$

$$\lim_{\lambda T \to \infty} Y_{2F} = R_{2F} \epsilon \tag{148}$$

$$\lim_{\lambda T \to \infty} Y_{3F} = R_{3F} \epsilon^2 \tag{149}$$

and the dependence of Y_{2F} and Y_{3F} on the duration of the time interval T is shown in Fig. 11.

As an example, consider a 4.48 kg ball of α-phase Pu comprised of 94% ^{239}Pu and 6% ^{240}Pu with $k_{\text{eff}} = 0.77$ and $M_{\text{e}} = 3.23$. The induced fission neutron multiplicity distribution for ^{239}Pu is shown in Fig. 12 and the spontaneous fission neutron multiplicity distribution for ^{240}Pu is shown in Fig. 13. The approximation is made that ^{239}Pu does not undergo spontaneous fission and

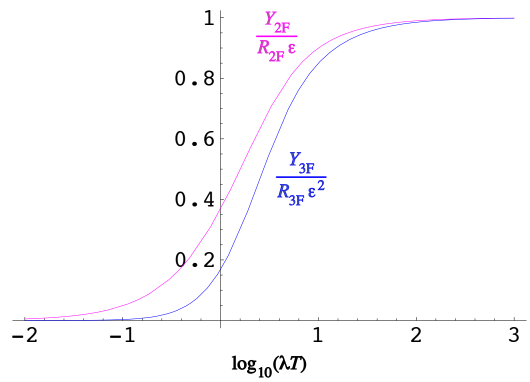

Figure 11: Dependence of Y_{2F} (purple) and Y_{3F} (blue) on the duration of the time interval T.

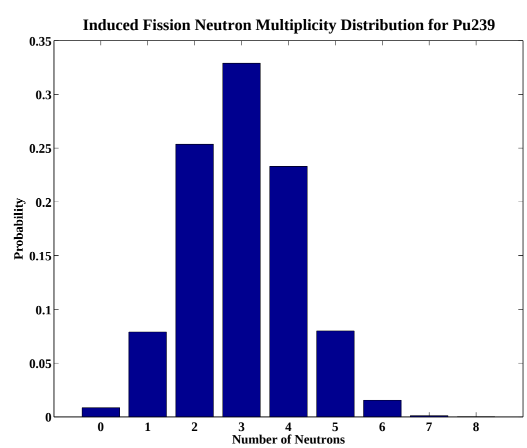

Figure 12: Induced fission neutron multiplicity distribution for ^{239}Pu.

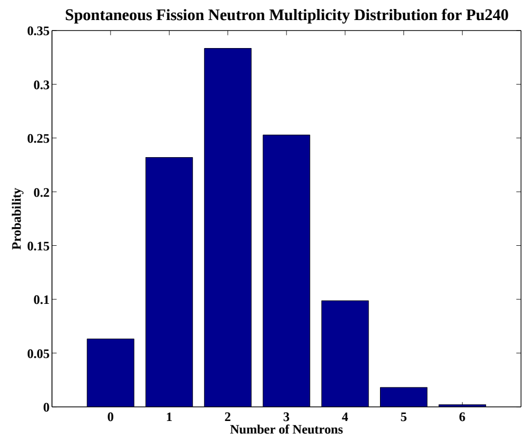

Figure 13: Spontaneous fission neutron multiplicity distribution for ^{240}Pu.

^{240}Pu does not undergo induced fission. Thus, no fission chains are initiated by induced fission and $F_{\mathrm{I}} = 0$. ^{240}Pu produces spontaneous fission neutrons at the rate of 1020 g$^{-1}s^{-1}$, making $F_{\mathrm{S}} = 1.26 \times 10^5\ s^{-1}$ for this example.

Suppose the bare Pu ball is placed near a neutron detector along with an amount of moderator such that $\lambda^{-1} = 40\ \mu$s, and the detector is allowed to count for 100 ms. The 100 ms total counting time is then divided into 100 ms/T time gates with $T = 1, 2, 3 \cdots 512\ \mu$s and the probability distributions $b_n(T)$ are determined for each T. The probability distribution for $T = 10\ \mu$s is shown in Fig. 14. For each $b_n(T)$, $Y_{2F}(T)$ and $Y_{3F}(T)$ are computed. The time dependence of these values follow the functional forms in Eq. 142 and Eq. 143, respectively, and can be fit to extract λ, $R_{2F}\epsilon$ and $R_{3F}\epsilon^2$. This is shown in Figs. 15 and 16 respectively. Also shown on these plots are the true values of $Y_{2F}(T)$ and $Y_{3F}(T)$ calculated using Eq. 130 from the moments of the neutron multiplicity distributions in Figs. 12 and 13, and the values listed above for F_{I}, F_{S}, and M_{e}.

If the efficiency of the detector ϵ is known a priori, and if reasonable assumptions about F_{I}, F_{S}, and the neutron multiplicity distributions (e.g. Fig. 12 and/or 13) can be made, then the fits to $Y_{2F}(T)$ and $Y_{3F}(T)$ in Figs. 15 and 16 and consequent values for $R_{2F}\epsilon$ and $R_{3F}\epsilon^2$ can be used to estimate the multiplication by solving Eq. 130 for M_{e}. The fitted value for λ also affords some insight as to the amount of moderation present; larger values for the neutron lifetime, λ^{-1}, tend to indicate more moderation.

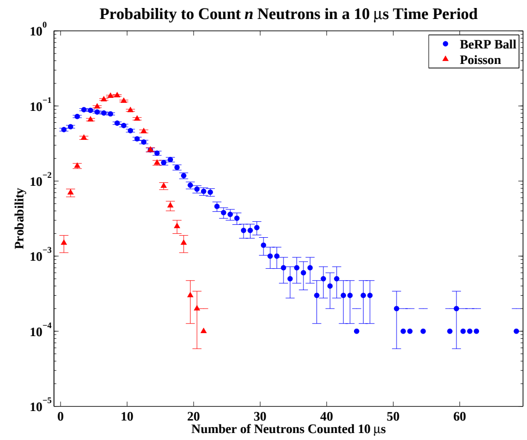

Figure 14: The probability distribution b_n with $T = 10\ \mu$s for a 100% efficient neutron detector (i.e. $\epsilon = 1$) near the 4.48 kg Pu Ball. There is moderator present such that $\lambda^{-1} = 40\ \mu$s. The detector was allowed to count for 100 ms. A Poisson distribution with the same count rate is shown for comparison.

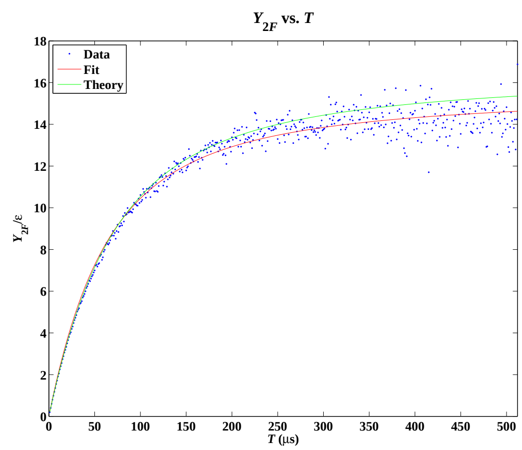

Figure 15: Y_{2F}/ϵ as a function of T from 1 μs to 512 μs. For each point, the 100 ms total counting time was divided into 100 ms/T time gates and the probability distribution $b_n(T)$ was determined for each T. Y_{2F} was then computed from each $b_n(T)$ (Data). The data was fit to determine λ and $R_{2F}\epsilon$ (Fit). The true value (Theory) is shown for comparison.

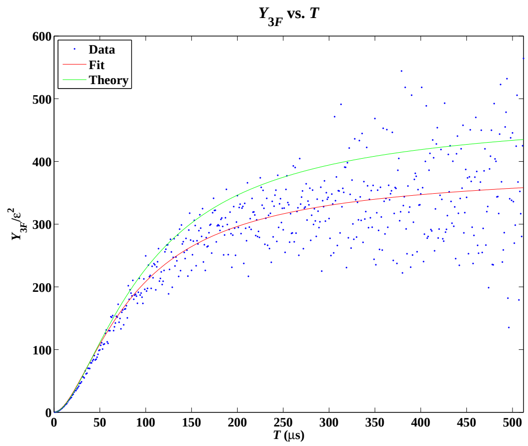

Figure 16: Y_{3F}/ϵ^2 as a function of T from 1 μs to 512 μs. For each point, the 100 ms total counting time was divided into 100 ms/T time gates and the probability distribution $b_n(T)$ was determined for each T. Y_{3F} was then computed from each $b_n(T)$ (Data). The data was fit to determine λ and $R_{3F}\epsilon^2$ (Fit). The true value (Theory) is shown for comparison.

10 FURTHER READING

The following references are useful in developing an understanding of the statistical theory of fission chains.

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